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LETTER TO THE EDITOR

The general epidemic process in a finite environment

H K Janssen, B Schaub and B Schmittmann

Institut für Theoretische Physik III, Universität Düsseldorf, Universitätsstrasse 1, D-4000 Düsseldorf 1, Federal Republic of Germany

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Abstract. The general epidemic process (GEP), a stochastic multiparticle process which exhibits a critical point near an absorbing state and leads to percolating clusters, is studied in a finite environment. Using renormalisation group techniques, we calculate the linear relaxation time in a cubic geometry of finite size L, with periodic boundary conditions imposed. The corresponding scaling behaviour to $O(\varepsilon)$ ($\varepsilon = 6-d$, d being the spatial dimension) is presented in universal form.

Epidemiology and ecology have become major research areas of all natural sciences within the last decade. It has been generally realised that most fundamental processes in these fields can be studied using the modern tools of statistical mechanics. In particular, universal (critical) properties of systems involving an infinite number of degrees of freedom can be successfully explored using renormalisation group (RG) ideas. Among a variety of techniques based upon these ideas, the powerful ε expansion provides one of the most reliable methods to determine critical exponents and scaling functions. It allows the analytic calculation of universal properties by means of a systematic expansion in powers of $\varepsilon = d_c - d$ around the upper critical dimension d_c.

So far, its main limitation consisted in the fact that it could be applied to systems of infinite size only. Systems of finite size, however, are essential for all numerical techniques which simulate processes on small samples and then extrapolate data using finite-size scaling ideas [1, 2]. To overcome this gap between analytic and numerical techniques, Brézin and Zinn-Justin [3] devised an analytic scheme which allows for the calculation of size-dependent universal scaling functions within an ε expansion.

In this letter, we will report results of a study of finite-size effects in a model of general epidemic processes (GEP) which exhibits a critical point near an absorbing state and leads to percolating clusters. The GEP is a stochastic multiparticle process which describes the temporal evolution of a local density of infected individuals, $n(x, t)$, where $x = (x_1, \ldots, x_d)$. It is characterised by the following features.

(i) There is an absorbing set of stationary states at $n(x, t) = 0$, corresponding to the situation where the epidemic has become extinct.

(ii) The disease spreads (diffusively) in the available environment.

(iii) Individuals can become immune to the disease. Thus, the net infection rate depends on the number of infected individuals *and* on the number of immune individuals, introducing a memory term into the process.

(iv) Microscopic degrees of freedom are subsumed in the form of a Langevin noise which, however, must respect the absorbing state. Hence, its correlations have to vanish for $n(x, t) = 0$.

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A Langevin equation which incorporates these characteristics in a simple yet self-contained form (in the renormalisation group spirit) is given by

$$\partial_t n(x,t) = \lambda_0 \Delta n(x,t) - \lambda_0 \tau_0 n(x,t) - \tfrac{1}{2}\lambda_0^2 w_0 n(x,t) \int_{-\infty}^{t} \mathrm{d}t'\, n(x,t') + \xi(x,t). \tag{1}$$

The first term on the right-hand side models the diffusive spreading (ii) of the disease. The next two terms represent the net infection according to (iii). $\xi(x, t)$ is a Langevin force, with correlations subject to (iv):

$$\langle \xi(x,t)\xi(x',t')\rangle = \lambda_0 w_0' n(x,t)\delta(x-x')\delta(t-t') \tag{2}$$

and λ_0, τ_0 and w_0 are constant couplings. By a simple rescaling we can ensure $w_0' = w_0$ which is in fact preserved under the renormalisation group. The characteristic feature of the GEP which distinguishes it from other evolution processes resides in the non-Markovian nature of the non-linear coupling associated with w_0. The term $\int_{-\infty}^{t} \mathrm{d}t'\, n(x, t')$ sums up the total density of individuals who catch the disease at any time between its outbreak and the time t. Thus it is a measure for the local density of immune individuals.

In the context of ecology, $n(x, t)$ may be interpreted as the local density of, say, a forest fire, where $\int_{-\infty}^{t} \mathrm{d}t'\, n(x, t')$ is proportional to the amount of burnt fuel (ash) at time t.

The process (1) and (2) was introduced in [4, 5] and studied in [6-9]. It was shown to exhibit a second-order phase transition near the absorbing state. Its critical properties have been analysed within an ε expansion about the upper critical dimension $d_c = 6$ [8]. In [8], it was also found that the GEP belongs to the universality class of dynamic percolation. Further, it was pointed out that even a possible reinfection of 'immunes' will not change the universal properties of (1) and (2), since reinfection leads to a coupling which is irrelevant in the renormalisation group sense near $d_c = 6$.

To study critical properties of the GEP within the framework of renormalised field theory it is convenient to recast the Langevin equation (1) in conjunction with (2) as a dynamic function [10-12]

$$J[\tilde{n}, n] = \int \mathrm{d}^d x \int \mathrm{d}t\, \tilde{n}(x,t)\left[\partial_t + \lambda_0\left(\tau_0 - \Delta - \tfrac{1}{2} w_0 \tilde{n}(x,t) + \lambda_0 w_0 \int_{-\infty}^{t} \mathrm{d}t'\, n(x,t')\right)\right] n(x,t). \tag{3}$$

Within this formalism all correlation and response functions can be expressed as functional averages with weight $\exp(-J)$. Finite-size effects can be investigated by considering the model (3) in a finite cubic geometry of linear size L with periodic boundary conditions imposed. We expand the fields n, $\tilde{n}$ in Fourier modes

$$n(x,t) = \sum_q \exp(\mathrm{i}qx) n(q,t)$$

in which each component of $q = (q_1, \ldots, q_d)$ takes only discrete values which are multiples of $2\pi/L$. It is obvious that (for $L \to \infty$) the $q = 0$ mode cannot be treated perturbatively as the critical point $\tau_0 = 0$ is approached, because the propagator of the $\tilde{n}$, n fields has an isolated pole at $q = 0$ and diagrams involving $\langle \tilde{n} n \rangle$ loops will diverge. Therefore, in order to calculate finite-size effects, one has to construct an effective action for the $q = 0$ mode (which itself has to be treated non-perturbatively) by tracing out all modes with $q \neq 0$. Details of this procedure, presented briefly below, can be found in [13], where we have discussed a stochastic model describing an evolution process related to (3) with a different upper critical dimension, of $d_c = 4$.

Introducing the decomposition

$$n(x,t) = \Phi(t) + \Psi(x,t) \qquad \Phi(t) = L^{-d}\int d^dx\, n(x,t) \tag{4}$$

$$\tilde{n}(x,t) = \tilde{\Phi}(t) + \tilde{\Psi}(x,t) \qquad \tilde{\Phi}(t) = L^{-d}\int d^dx\, n(x,t) \tag{5}$$

which allows the separate treatment of the $q = 0$ modes $\Phi(t)$, $\tilde{\Phi}(t)$ in $J[\tilde{n}, n]$, we obtain

$$J = J^{(0)}_{\mathrm{hom}} + J^{(2)} + \dots \tag{6}$$

with $J^{(0)}_{\mathrm{hom}}$ being the free (0) functional for the $q = 0$ (homogeneous) modes

$$J^{(0)}_{\mathrm{hom}} = L^d \int dt\, \tilde{\Phi}\left[\partial_t + \lambda_0\left(\tau_0 - \tfrac{1}{2}w_0\tilde{\Phi} + w_0\lambda_0\int_{-\infty}^{t} dt'\,\Phi(t')\right)\right]\Phi \tag{7}$$

and

$$J^{(2)} = \int dt \int d^dx\left[\tilde{\Psi}\left(\partial_t + \lambda_0(\tau_0 - \Delta) - \lambda_0 w_0\tilde{\Phi} + \lambda^2 w\int_{-\infty}^{t} dt'\,\Phi(t')\right)\Psi\right.$$
$$\left. + \lambda_0 w_0\left(\tilde{\Psi}\Phi\lambda_0\int_{-\infty}^{t} dt'\,\Psi(t') + \tilde{\Phi}\Psi\lambda_0\int_{-\infty}^{t} dt'\,\Psi(t') - \tfrac{1}{2}\tilde{\Psi}^2\Phi\right)\right]. \tag{8}$$

Integration over all $q \neq 0$ modes can now be performed perturbatively via

$$\exp(-J_{\mathrm{hom}}) = \exp(-J^{(0)}_{\mathrm{hom}})\int [d\tilde{\Psi}][d\Psi]\exp(-J^{(2)}) \tag{9}$$

and will lead to an effective action J_{hom}. This calculation corresponds to a double expansion in powers of the fields $\tilde{\Phi}$ and Φ, arising from the vertices in $J^{(2)}$, and in the number of loops, due to the insertion of those vertices which do not contain the fields Φ, $\tilde{\Phi}$ in diagrams with a fixed number of $\tilde{\Phi}$ and Φ fields. Note that we have omitted these vertices in (6), since they do not contribute at the one-loop level.

To this order in the number of loops and to third order in Φ, $\tilde{\Phi}$ we can express $J_{\mathrm{hom}}[\tilde{\Phi}, \Phi]$ in the following form (see [13] for details):

$$J_{\mathrm{hom}}[\tilde{\Phi}, \Phi] = L^d\int dt\left(\hat{r}_0\tilde{\Phi}\dot{\Phi} + \lambda_0\hat{\tau}_0\tilde{\Phi}\Phi - \tfrac{1}{2}\lambda_0\hat{w}_0\tilde{\Phi}^2\Phi + \lambda_0^2\hat{w}_0\tilde{\Phi}\Phi\int_{-\infty}^{t} dt'\,\Phi(t')\right) \tag{10}$$

where

$$\hat{r}_0 = 1 - \tfrac{3}{4}bw_0^2 \qquad \hat{\tau}_0 = \tau_0 + \tfrac{1}{2}aw_0^2 \qquad \hat{w}_0 = w_0(1 - 2bw_0^2) \tag{11}$$

with

$$a(\tau_0, L) = L^{-d}\sum_{q\neq 0}\frac{1}{(\tau_0 + q^2)^2} \tag{12a}$$

$$b(\tau_0, L) = L^{-d}\sum_{q\neq 0}\frac{1}{(\tau_0 + q^2)^3}$$
$$= -\frac{1}{2}\frac{d}{d\tau_0}a(\tau_0, L). \tag{12b}$$

The sums in (12a) and (12b) are evaluated in the appendix:

$$a(\tau, L) = G_\varepsilon (L/2\pi)^{6-d}(-2/\varepsilon + \sigma(x)/x) \tag{13a}$$

$$b(\tau, L) = G_\varepsilon (L/2\pi)^{6-d}(1/\varepsilon - \tfrac{1}{2}\sigma'(x)) \tag{13b}$$

with $x = \tau(L/2\pi)^2$ and $G_\varepsilon = \Gamma(1+\frac{1}{2}\varepsilon)/(4\pi)^{d/2}$. In (13$a$) and (13$b$) we have isolated the pole terms and $\sigma(x)$ ($\sigma'(x) = \mathrm{d}\sigma(x)/\mathrm{d}x$) is an analytic function for $x > -1$. Its limiting behaviour for $x \gg 1$ (see the appendix) is given by

$$\sigma(x) = x(\ln x - 1) + \text{constant}. \tag{14}$$

From the form of the bare action, equation (10), and the one-loop corrections, we can see that the partial trace over the $q \neq 0$ modes generates ε poles *and* shifts in the couplings. The ε poles can be absorbed in the well known bulk Z factors, presented in [8], thus replacing the bare couplings and fields λ_0, τ_0, w_0, Φ, $\tilde{\Phi}$ by their renormalised counterparts λ, τ, w, Φ_R, $\tilde{\Phi}_R$. The shifts are taken care of by a finite redefinition of the coupling constants and the renormalised action (including the one-loop corrections) for the $q = 0$ modes can thus be written

$$J_{\text{hom}} = L^d \int \mathrm{d}t \left(\hat{r}\tilde{\Phi}_R\dot{\Phi}_R + \lambda\hat{\tau}\tilde{\Phi}_R\Phi_R - \tfrac{1}{2}\lambda\hat{w}\tilde{\Phi}_R^2\Phi_R + \lambda^2\hat{w}\tilde{\Phi}_R\Phi_R \int_{-\infty}^{t} \mathrm{d}t'\Phi_R(t') \right) \tag{15}$$

where

$$\hat{r} = 1 + \tfrac{3}{8}u[\sigma'(x) - 2\ln(\mu L/2\pi)] \tag{16a}$$

$$\hat{\tau} = \tau\{1 + \tfrac{1}{2}u[\sigma(x)/x - 2\ln(\mu L/2\pi)]\} \tag{16b}$$

$$\hat{w} = w\{1 + u[\sigma'(x) - 2\ln(\mu L/2\pi)]\} \tag{16c}$$

and we have introduced a new coupling $u = G_\varepsilon \mu^{-\varepsilon} w^2$.

It is a generic feature of the technique employed here that for $d < d_c$ the shifts produced by the loop corrections can be absorbed by a redefinition of the couplings. This is analogous to the calculation in the purely static case (Φ^4 theory with $d_c = 4$) and has been reported in [14] for models with simple relaxational dynamics. Note, however, that in the present model, similar to the model studied in [13], we need a non-trivial wavefunction renormalisation already at one-loop order, and thus had to introduce an additional shift (equation (16a)). It should also be noted, following from equations (12) and (16), that for $d > 6$ the loop corrections which result from tracing out the $q \neq 0$ modes are irrelevant. They only produce a finite shift in the critical parameter τ and a finite renormalisation of the couplings.

Now we can determine the linear relaxation time for our model in a finite geometry. This characteristic time corresponds to the smallest non-zero eigenvalue of the Fokker–Planck equation associated with the Langevin equation (1) and can be derived from a simple rescaling of the dynamic functional J_{hom}. With

$$\Phi(t) = \alpha\varphi(s) \qquad \tilde{\Phi}(t) = \tilde{\alpha}\tilde{\varphi}(s) \qquad t = \beta s \tag{17}$$

and utilising a symmetry transformation of the dynamic functional J observed in [8] which amounts to $\beta\lambda = \tilde{\alpha}/\alpha$ we can write

$$J_{\text{hom}} = L^d \int \mathrm{d}s \left(\hat{r}\tilde{\alpha}\alpha\tilde{\varphi}\dot{\varphi} + \tilde{\alpha}^2\hat{\tau}\tilde{\varphi}\varphi - \tfrac{1}{2}\tilde{\alpha}^3\hat{w}\tilde{\varphi}^2\varphi + \tilde{\alpha}^3\hat{w}\tilde{\varphi}\varphi \int_{-\infty}^{s} \mathrm{d}s'\,\varphi(s') \right) \tag{18}$$

where we have dropped the index R on the fields for simplicity. Choosing now α, $\tilde{\alpha}$ such that

$$L^d\tilde{\alpha}\alpha\hat{r} = L^d\tilde{\alpha}^3\hat{w} = 1 \tag{19}$$

and defining γ via

$$\tilde{\alpha}^2\hat{\tau}L^d = \hat{\tau}L^d(\hat{w}L^d)^{-2/3} = \gamma \tag{20}$$

we obtain

$$J_{\text{hom}} = \int \mathrm{d}s\left(\tilde{\varphi}\dot{\varphi} + \gamma\tilde{\varphi}\varphi - \tfrac{1}{2}\tilde{\varphi}^2\varphi + \tilde{\varphi}\varphi\int_{-\infty}^{s}\mathrm{d}s'\,\varphi(s')\right). \tag{21}$$

The essential feature of (21) is the dependence of the whole dynamical functional on one single parameter only.

Therefore, the linear relaxation time S_R, measured in the rescaled time S, must be a function of this single parameter only:

$$S_\mathrm{R} = f_\mathrm{R}(\gamma) \tag{22}$$

whence

$$t_\mathrm{R} = \beta S_\mathrm{R} = \frac{\hat{r}}{\lambda\hat{w}}(\hat{w}L^d)^{1/3}f_\mathrm{R}[(\hat{\tau}/\hat{w})(\hat{w}L^d)^{1/3}]. \tag{23}$$

Note that the scaling function f in (23) is associated with the effective action for the $q=0$ modes and cannot be computed perturbatively. It may however be derived from the Fokker-Planck equation corresponding to (21).

Using (16) and (23) we obtain the linear relaxation time at the fixed point $u_* = \frac{2}{7}\varepsilon$ [8] in universal form:

$$t_\mathrm{R} = (\lambda\mu^2)^{-1}(\mu L/2\pi)^z h_\mathrm{R}(y)f_\mathrm{R}(g(y)) \tag{24a}$$

$$h_\mathrm{R}(y) = (1+y)^{(d/3-z)\nu}\{1+\tfrac{1}{12}\varepsilon[\ln(1+y)-\sigma'(y)]+\mathrm{O}(\varepsilon^2)\} \tag{24b}$$

$$g(y) = y(1+y)^{d\nu/3-1}\{1+\tfrac{1}{21}\varepsilon[\ln(1+y)+3\sigma(y)/y-4\sigma'(y)]+\mathrm{O}(\varepsilon^2)\} \tag{24c}$$

$$y = \tau\mu^{-2}(\mu L/2\pi)^{1/\nu}$$

with $\nu = \frac{1}{2}+\frac{5}{84}\varepsilon$ and $z = 2-\frac{1}{6}\varepsilon$ being the critical exponents to $\mathrm{O}(\varepsilon)$. In the bulk limit $y\to\infty$, t_R must become independent of L, a fact which follows directly from the limiting behaviour of the scaling function [13]:

$$f_\mathrm{R}(x) \sim 1/x \qquad \text{for } x\to\infty. \tag{25}$$

The other limit of interest, $y\to 0$, corresponds to a finite system at the critical 'temperature' of the bulk. Equation (24) then yields

$$t_\mathrm{R} = (\lambda\mu^2)^{-1}(\mu L/2\pi)^z h_\mathrm{R}(0)f_\mathrm{R}(g(0)). \tag{26}$$

The limiting forms of the functions $h_\mathrm{R}(y)$, $g(y)$ for $y\to 0$ are given by (see the appendix)

$$h_\mathrm{R}(0) = 1+0.1907\varepsilon \tag{27}$$

$$g(0) = -0.0489\varepsilon. \tag{28}$$

It follows from our results that whereas finite-size scaling is valid below $d_\mathrm{c}=6$, it is not valid for $d\geqslant 6$, since there is a singularity for $\varepsilon\to 0$ [15, 16].

To summarise, we have calculated the linear relaxation time for the GEP in a finite geometry. As epidemic processes under realistic conditions are always restricted to finite environments, we expect our results to be important for simulations of small samples in two and three dimensions (for a simulation of a large GEP sample in $d=2$ see [6]). Furthermore, as the GEP belongs to the class of systems exhibiting non-equilibrium critical behaviour, we believe that further studies of the GEP and related processes [13] will be of great interest.

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Appendix

In this appendix we derive some important properties of the function $\sigma(x)$. The lattice sum to be evaluated is given by

$$a(\tau, L) = L^{-d} \sum_{q\neq 0} \frac{1}{(\tau+q^2)^2} = \left(\frac{L}{2\pi}\right)^{4-d} \frac{1}{(2\pi)^d} \sum_{n\neq 0} \frac{1}{(x+n^2)^2} \tag{A1}$$

where $x=\tau(L/2\pi)^2$ and $\boldsymbol{n}=(n_1,\ldots,n_d)$, $n_i=$ integer. The sum in (A1) can be converted into an integral

$$I(x) = \frac{1}{(2\pi)^d} \sum_{n\neq 0} \frac{1}{(x+n^2)^2} = \frac{1}{(2\pi)^d} \int_0^\infty \mathrm{d}t [A(t)^d - 1] t\, \mathrm{e}^{-tx} \tag{A2}$$

with

$$\begin{aligned} A(t) &= \sum_{n=-\infty}^{\infty} \mathrm{e}^{-tn^2} = 1 + 2\mathrm{e}^{-t} + \ldots \\ &= (\pi/t)^{1/2} A(\pi^2/t) = (\pi/t)^{1/2}[1 + 2\exp(-\pi^2/t) + \ldots]. \end{aligned} \tag{A3}$$

For $x \gg 1$ the integral may be obtained using dimensional regularisation:

$$\begin{aligned} I(x) &= I(0) + \frac{\Gamma(2-\frac{1}{2}d)}{(4\pi)^{d/2}} x^{d/2-2} + \frac{1}{(2\pi)^d} \int_0^\infty \mathrm{d}t \left[A(t)^d - 1 - \left(\frac{\pi}{t}\right)^{d/2} \right] t(\mathrm{e}^{-tx} - 1) \\ &= -\frac{2G_\varepsilon}{\varepsilon(1-\frac{1}{2}\varepsilon)} x^{1-\varepsilon/2} + \text{constant} + \mathrm{O}(1/x) \end{aligned} \tag{A4}$$

with $G_\varepsilon = \Gamma(1+\frac{1}{2}\varepsilon)/(4\pi)^{d/2}$. Thus we define a function $\sigma(x)$ by

$$I(x) = G_\varepsilon[-(2/\varepsilon)x + \sigma(x) + \mathrm{O}(\varepsilon)] \tag{A5}$$

where

$$\sigma(x) = x(\ln x - 1) + \text{constant} + \mathrm{O}(1/x) \tag{A6}$$

for $x \gg 1$.

In order to find an expansion for $\sigma(x)$ if $|x| \ll 1$ we expand $I(x)$ in powers of x:

$$I(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \frac{1}{(2\pi)^d} \int_0^\infty \mathrm{d}t\, t^{k+1} [A(t)^d - 1]. \tag{A7}$$

The integral in (A7) is convergent in $d=6$ dimensions if $k\geqslant 2$ and can be calculated, using the identity given in [17]:

$$\int_0^\infty \mathrm{d}t\, t^\lambda[A(t)^6-1]=4\Gamma(\lambda+1)[4\zeta(\lambda-1)\beta(\lambda+1)-\beta(\lambda-1)\zeta(\lambda+1)] \tag{A8}$$

where ζ denotes Riemann's ζ function and β is defined by the sum

$$\beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z}. \tag{A9}$$

The remaining two integrals ($k=0,1$) can be performed with the help of a suitable analytic continuation.

For $k=1$ we write

$$\begin{aligned}
&\frac{1}{(2\pi)^d}\int_0^\infty \mathrm{d}t\, t^2[A(t)^d-1]\\
&\quad=\frac{1}{(2\pi)^d}\int_0^\infty \mathrm{d}t\, t^2[A(t)^d-1-(\pi/t)^{d/2}\,\mathrm{e}^{-t}]+\frac{\Gamma(3-\frac{1}{2}d)}{(4\pi)^{d/2}}\\
&\quad=\frac{2G_\varepsilon}{\varepsilon}+\lim_{\delta\to 0}\frac{1}{(2\pi)^6}\int_0^\infty \mathrm{d}t\, t^{2+\delta}\{[A(t)^6-1]-(\pi/t)^3\,\mathrm{e}^{-t}\}+\mathrm{O}(\varepsilon)\\
&\quad=G_\varepsilon\{2/\varepsilon+C_{\mathrm{E}}+\tfrac{3}{2}+(32/\pi^3)\beta'(3)-2\zeta(3)/\pi^2\}+\mathrm{O}(\varepsilon)
\end{aligned} \tag{A10}$$

where we have used (A8). C_{E} is Euler's constant, and the prime denotes the derivative. For $k=0$ we write

$$\begin{aligned}
&\frac{1}{(2\pi)^d}\int_0^\infty \mathrm{d}t\, t[A(t)^d-1]\\
&\quad=\frac{1}{(2\pi)^d}\int_0^\infty \mathrm{d}t\, t[A(t)^d-1-(\pi/t)^{d/2}(1+t)\,\mathrm{e}^{-t}]+\frac{\Gamma(4-\frac{1}{2}d)}{(4\pi)^{d/2}(2-\frac{1}{2}d)}\\
&\quad=\lim_{\delta\to 0}\frac{1}{(2\pi)^6}\int_0^\infty \mathrm{d}t\, t^{1+\delta}[A(t)^6-1-(\pi/t)^3(1+t)\,\mathrm{e}^{-t}]-\frac{1}{(4\pi)^3}+\mathrm{O}(\varepsilon)\\
&\quad=-G_\varepsilon\left(\frac{8\beta(2)}{\pi^3}+\frac{1}{3\pi}\right)+\mathrm{O}(\varepsilon)
\end{aligned} \tag{A11}$$

where we have again used (A8). From the definitions (A5) and (A7) we obtain

$$\sigma(x)=\sum_{k=0}^{\infty}\sigma_k(-x)^k \tag{A12}$$

where

$$\begin{aligned}
&\sigma_0=-\left(\frac{8\beta(2)}{\pi^2}+\frac{1}{3\pi}\right)\qquad \sigma_1=C_{\mathrm{E}}+\tfrac{3}{2}+32\beta'(3)/\pi^3-2\zeta(3)/\pi^2\\
&\sigma_k=[4(k+1)/\pi^3][4\zeta(k)\beta(k+2)-\beta(k)\zeta(k+2)]\,k\geqslant 2.
\end{aligned} \tag{A13}$$

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